Solutions to Hand-In Assignment 4

1. Let G be an open subset of (M, d) and F be a closed subset of (M, d). Prove or disprove: G\F is an open subset of (M, d). [10 pts]

Solution: The set G\F = $\{x \in M: x \in G \text{ } and \text{ } x \notin F\}$ can also be written as $G \cap F^c$, which is the intersection of two open sets. Hence G\F is an open subset of (M, d).

2. Let G = $\{(x, y) \in \mathbb{R}^2 : x \neq y\}$. Prove that G is an open subset of $\mathbb{R}^2$. [10 pts]

Solution: Consider the function $f: \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x - y$. This function is a polynomial in two variables (and even a linear map) and is therefore continuous. Notice that the set G can be expressed as the intersection $\{(x, y) \in \mathbb{R}^2: f(x, y) < 0\} \cup \{(x, y) \in \mathbb{R}^2: f(x, y) > 0\} = f^{-1}(-\infty, 0) \cup f^{-1}(0, \infty)$. This is a union of open sets which must be open.

3. Suppose that E is a nowhere dense subset of (M, d). Prove or disprove: E^c is everywhere dense in M. [10 pts]

Solution: If E is nowhere dense, then in particular $E^\circ = \emptyset$. This implies that the intersection of E^c with any neighborhood in M is nonempty. Thus every element of M is either a point of E^c or a cluster point (i.e. limit point) of E^c. Thus $\overline{E^c}$ = M and therefore E^c is dense in M.

4. Let ◊ be a set constructed out of the interval [0, 1] as follows:

   Step 1: Partition the interval into 5 parts of equal length and remove every other (open) part. Thus, you obtain the set
   $I_1 = [0, 1/5] \cup [2/5, 3/5] \cup [4/5, 1]$

   Step 2: Partition each of the interval segments of I_1 further into 5 parts of equal length and remove every other (open) part. Thus, you obtain the set
   $I_2 = [0, 1/25] \cup [2/25, 3/25] \cup [4/25, 1/5] \cup$
   $\cup [2/5, 11/25] \cup [12/25, 13/25] \cup [14/25, 3/5] \cup$
   $\cup [4/5, 21/25] \cup [22/25, 23/25] \cup [24/25, 1]$

   Step n: Partition each of the interval segments of I_{n-1} into 5 parts of equal length and remove every other (open) part to obtain I_n.

Set $\lozenge = \bigcap_{n=1}^{\infty} I_n$.

a) Is $\lozenge$ closed as a subset of $\mathbb{R}$? [2 pts]

Solution: The complement of $\lozenge$ is a union of open intervals. Therefore $\lozenge$ is a closed set.

b) Is $\lozenge$ countable or uncountable? [2 pts]

Solution: $\lozenge$ is uncountable

c) Is $\lozenge$ dense, nowhere dense, or neither? [2 pts]

Solution: $\lozenge$ is nowhere dense.

d) Is $\lozenge$ perfect? [2 pts]

Solution: $\lozenge$ is perfect.

e) What is the "length" (i.e measure) of $\lozenge$? [2 pts]

Solution: The length of $\lozenge$ is 1 - length of $\lozenge^c$. Where the complement is taken relative to the interval [0, 1]. Thus, it is $1 - \sum_{n=1}^{\infty} 2\frac{3^{n-1}}{5^n} = 1 - \frac{2}{5}\frac{1}{1-3/5} = 1 - 1 = 0$.

f) The elements of $\lozenge$ can be most easily described in terms of some base p decimal expansion. What p should we choose? In terms of the decimal expansion base p, how would you decide whether x is an element of $\lozenge$? [10 pts]

Solution: In each step of the construction of $\lozenge$, the intervals are partitioned into 5 equal parts. This suggests that the elements of $\lozenge$ are easiest to describe in base 5. By inspecting the constructed intervals in each step, it is easily seen that x is an element of $\lozenge$ if and only if x can be written in base 5 as x = $0.(2a_1)(2a_2)\ldots(2a_n)\ldots$ where each a_n= 0, 1, or 2.

g) Construct a Cantor-like function for $\lozenge$. [5 pts]

Solution: Define $f\colon \lozenge \to [0,1]$ by $f(x) = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$, whenever $x = \sum_{n=1}^{\infty} \frac{2a_n}{5^n}$ (where each $a_n = 0, 1, \text{or } 2$.